An improvement of de Jong–Oort's purity theorem

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Abstract. Consider an F-crystal over a noetherian scheme S. De Jong–Oort's purity theorem states that the associated Newton polygons over all points of S are constant if this is true outside a subset of codimension bigger than 1. In this paper we show an improvement of the theorem, which says that the Newton polygons over all points of S have a common break point if this is true outside a subset of codimension bigger than 1.

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1. introduction

De Jong–Oort's purity theorem [1, Theorem 4.1] states that for an F-crystal over a noetherian scheme S of characteristic p the associated Newton polygons over all points of S are constant if this is true outside a subset of codimension bigger than 1. This theorem has been strengthened and generalized by Vasiu [6], who has shown that each stratum of the Newton polygon stratification defined by an F-crystal over any reduced, not necessarily noetherian $\mathbb{F}_p$-scheme S is an affine S-scheme. In the case of a family of p-divisible groups, alternative proofs of the purity have been given by Oort [15] and Zink [16]. In this paper we show an improvement, which implies that for an F-crystal over a noetherian scheme S the Newton polygons over all points have a common break point if this is true outside a subset of codimension bigger than 1. As to a stronger statement analogous to that in Vasiu's paper, our method does not apply. The main result is the following theorem.

THEOREM 1.1. *Let S be a locally Noetherian scheme of characteristic p and $\mathcal{E}$ be an F-crystal over S. Fix $s \in S$. If there exists an open neighborhood U of s in S such that the Newton polygons $NP(\mathcal{E})_x$ over all points $x \in U \backslash \{s\}$ have a common break point, then either codim$(\{s\}^-, U) \leq 1$ or $NP(\mathcal{E})_s$ has the same break point.*

The following example explains how Theorem 1.1 improves de Jong-Oort purity theorem. Look at *Figure* 1. Consider the spectrum of some local Noetherian integral domain of dimension 2 and characteristic p. Then we ask: does there exist an F-crystal such that the associated Newton polygon over the closed point is ξ, over a finite number of points of codimension 1 is γ and over each of all other points is η? Theorem 1.1 tells us that the answer is negative, while it cannot be easily seen from [1, Theorem 4.1].
In our main theorem, the condition on "one of the break points" cannot be generalized to an arbitrary point of the Newton polygon which is not a break point. Consider a family of elliptic curves $f : \mathcal{X} \to S$, where S is a curve over a field k of positive characteristic. Look at *Figure* 2. Assume that all the fibers of f are ordinary except over a closed point $0 \in S$. Then *Figure* 2 shows all the Newton polygons associated to the family of abelian surfaces $\mathcal{X} \times_k \mathcal{X} \to S \times_k S$. Namely, over the special point $(0, 0)$ the associated Newton polygon is ξ; over each point in $\{0\} \times S \cup S \times \{0\}$, it is γ'; over each of all other points, it is η'. We see that outside the one-point set $\{(0, 0)\}$ of codimension 2, the Newton polygons have a common point P.

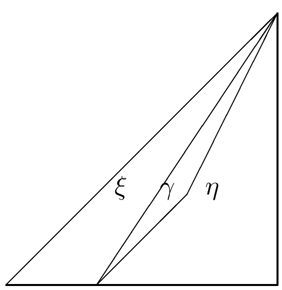

Figure 1

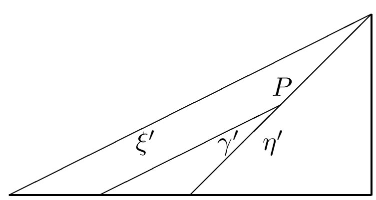

Figure 2

This paper is organized as follows. In Section 2, we review some facts about F-crystals before showing that if the Newton polygon of an F-crystal over a field has a break point $(1, m)$, then there exists a unique subcrystal of rank 1 and slope m. In Section 3, we describe the kernel of $\mathrm{Gal}(\overline{K}/K) \to \pi_1(X, \overline{\eta})$ as the normal subgroup generated by local kernels (Proposition 3.2) and particularly obtain another description when X is the spectrum of a discrete valuation ring (Corollary 3.9). In Section 4, we define the Galois representation associated to an F-crystal, and discuss the relationship between its ramification property and Newton slopes (see proposition 4.6). Section 5 contains the proof of Theorem 1.1. The proof essentially follows the proof of [1, Theorem 4.1], yet is

more accessible because the relationship between the ramification property of the representation and the Newton slopes has been clarified.

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2. Results on F-crystals

2.1. Conventions. In this paper, k always denotes a field of characteristic p, where p is a prime number; $\overline{k}$ denotes an algebraic closure of k; S denotes a connected scheme of characteristic p. We use the term *crystal* to mean a crystal of finite locally free $\mathcal{O}_{cris}-$modules. See [4, Page 226]. Here $\mathcal{O}_{cris}$ denotes the structure sheaf on the category CRIS(S/Spec $\mathbb{Z}_p$) (big crystalline site of S). If $T \to S$ is a morphism, then we use $\mathcal{E}|_T$ to denote the pullback of $\mathcal{E}$ to CRIS(T/Spec $\mathbb{Z}_p$). For a crystal $\mathcal{E}$, we denote by $\mathcal{E}^{(n)}$ the pullback of $\mathcal{E}$ by the nth iterate of the Frobenius endomorphism of S. An *F-crystal over S* is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a crystal over S and $F : \mathcal{E}^{(1)} \to \mathcal{E}$ is a morphism of crystals. We usually denote an F-crystal by $\mathcal{E}$, with the map F being understood. Recall that $\mathcal{E}$ is a *nondegenerate F-crystal* if the kernel and cokernel of F are annihilated by some power of p, see [11, 3.1.1]. All F-crystals in this paper will be nondegenerate.

A perfect scheme S in characteristic p is a scheme such that the Frobenius map $(-)^p : \mathcal{O}_S \to \mathcal{O}_S$ is an isomorphism. A crystal over a perfect scheme S is simply given by a finite locally free sheaf of $W(\mathcal{O}_S)$-modules (see [3, Page 141]).

2.2. Suppose that $S = \text{Spec } k$. Choose a Cohen ring Λ for k, and let $\sigma : \Lambda \to \Lambda$ be a lift of $Frobenius$ on k. By [5, Proposition 1.3.3], we know that an F-crystal $\mathcal{E}$ over k is given by a triple (M, ∇, F) over Λ, where M is a finite free Λ-module of rank r, ∇ is an integrable, topologically quasi-nilpotent connection, and F is a horizontal σ-linear self-map of M.

2.3. Let k^{pf} be the perfect closure of k. Note that under the identification $k^{pf} = \varinjlim(k \to k \to \ldots)$, and by [7, Chapter II, Prop 10], we obtain

$$W(k^{pf}) = p\text{-adic completion of } \varinjlim(\Lambda \xrightarrow{\sigma} \Lambda \xrightarrow{\sigma} \ldots).$$

Furthermore σ can be extended to an endomorphism of $W(k^{pf})$, which is a lift of Frobenius on k^{pf}, still denoted by σ. Thus we get an injection $\Lambda \to W(k^{pf})$ compatible with σ.

The pullback $\mathcal{E}|_{\text{Spec}(k^{pf})}$ of $\mathcal{E}$ corresponds to the pair $(M \otimes_\Lambda W(k^{pf}), F \otimes \sigma)$. According to [3, 1.3], we can describe Newton slopes associated to $\mathcal{E}|_{\text{Spec}(k^{pf})}$ as follows. Choose an algebraic closure $\overline{k}$ of k^{pf} and some positive integer N divisible by $r!$. Consider the valuation ring $R = W(\overline{k})[X]/(X^N - p) = W(\overline{k})[p^{1/N}]$

and denote its fraction field by K. We extend σ to an automorphism of R by requiring that $\sigma(X) = X$. Then by Dieudonné(cf [10]), $M \otimes_{W(\overline{k})} K$ admits a K-basis $e_1, \ldots, e_r$ such that $(F \otimes \sigma)(e_i) = p^{\lambda_i} e_i$ and $0 \leq \lambda_1 \leq \ldots \leq \lambda_r$. These r rational numbers are defined to be the *Newton slopes* of (M, F) or $\mathcal{E}$.
For each λ, we define $\mathrm{mult}(\lambda)$ as the number of times λ occurs among $\{\lambda_1, \ldots, \lambda_r\}$. By Dieudonné again, the product $\lambda\ \mathrm{mult}(\lambda) \in \mathbb{Z}_{\geq 0}$ for each λ. The Newton Polygon of (M, F) is a polygonal chain consisting of line segments $S_1, \ldots,\ S_r$, where S_i connects the two points $(i-1, \lambda_1 + \ldots + \lambda_{i-1})$ and $(i, \lambda_1 + \ldots + \lambda_i)$. The points at which the Newton polygon changes slope are called *break points*.
We now turn to an F-crystal $\mathcal{E}$ over an arbitrary $\mathbb{F}_p$-scheme S. For every point $s \in S$, let $s : \mathrm{Spec}\ k(s) \to S$ be the natural map. We can assign to s the Newton polygon associated with $\mathcal{E}|_{\ \mathrm{Spec}\ k(s)}$, denoted by $NP(S, \mathcal{E})_s$ or $NP(\mathcal{E})_s$.
The following result about the existence of some special subcrystal will be significant in proving the theorem.

PROPOSITION 2.4. *Let $(\mathcal{E}, F)$ be a crystal over $S = Spec(k)$. If the first break point of $NP(S, \mathcal{E})$ is $(1, m)$, where $m \in \mathbb{Z}_{\geq 0}$, then it has a unique subcrystal $\mathcal{E}_1 \subset \mathcal{E}$ of rank 1 and slope m.*

Proof: Let (M, ∇, F) be the triple corresponding to the crystal $\mathcal{E}$ by 2.2. Then the existence of the required subcrystal is equivalent to the existence of a unique Λ-submodule M_1 of rank 1 and slope m, preserved by the action of ∇. We will first find a submodule of rank 1 and slope m, then show it is preserved by ∇. The uniqueness of such a submodule follows from the fact that the lowest slope is of multiplicity 1.
Choose $\overline{k} \supset k^{pf} \supset k$. From 2.3, we have a faithfully flat homomorphism $\Lambda \xrightarrow{i} W(\overline{k})$. Let $\overline{M} = M \otimes_\Lambda W(\overline{k})$. By [3, Theorem 2.6.1], there is an isogeny $\psi : \overline{M} \to N$, where $\frac{1}{p^m} F_N : N \to N$ is a σ-linear self-map. By [3, Theorem 1.6.1], N has a unique free submodule N_1 of rank 1 and slope m such that N/N_1 is free as a $W(\overline{k})$-module. Let $\overline{M}_1 = \psi^{-1}(N_1)$. It is clear that $\overline{M}_1$ is a module of rank 1 and slope m, and $\overline{M}_2 = \overline{M}/\overline{M}_1$ is a free Λ-module of rank $r - 1$ and slopes $> m$.
Since ψ is an isogeny, there exists some $D \in \mathbb{Z}_{>0}$ such that $p^D \psi^{-1}(N) \subset \overline{M}$. As for every $\nu > 0$, $p^D (\frac{F}{p^m})^\nu = p^D \psi^{-1} (\frac{F_N}{p^m})^\nu \psi$, thus $p^D (\frac{F}{p^m})^\nu : \overline{M} \to \overline{M}$. Actually we can choose $D_\nu \in [0, D]$ such that the matrix of $\overline{f^\nu} = p^{D_\nu} (\frac{F}{p^m})^\nu$ mod p does not vanish. Let $f^\nu = p^{D_\nu} (\frac{F}{p^m})^\nu : M \to M$, then f^ν mod p does not vanish either. Since the Newton slopes of $\overline{M}_2$ are greater than m, according to [3, 1.4.3], for each $n > 0$ there exists $c_n > 0$ such that $\overline{f^\nu}(\overline{M}_2) \subset p^n \overline{M}_2$ for all $\nu \geq c_n$. Let $\overline{f^\nu_n} : \overline{M}/p^n\overline{M} \to \overline{M}/p^n\overline{M}$. Then $\mathrm{Im}(\overline{f^\nu_n}) \subset \overline{M}_1/p^n\overline{M}_1$. Let $\overline{E^\nu_n} =< \mathrm{Im}(\overline{f^\nu_n}) >$. Note that $< G >$ is denoted as the smallest R/p^nR-submodule of M containing G, where R is a discrete valuation ring with p as its uniformizer, M is a finite free R/p^nR-module and $G \subset M$ a subset.
Let $f^\nu_n : M/p^nM \to M/p^nM$, $E^\nu_n =< \mathrm{Im}(f^\nu_n) >$, and $E_n = \cap_{\nu \geq c_n} E^\nu_n$. We get $\overline{E^\nu_n} = E^\nu_n \otimes_\Lambda W(\overline{k})$, and $\overline{E}_n = E_n \otimes_\Lambda W(\overline{k}) = \cap_{\nu \geq c_n} \overline{E^\nu_n}$. By the above

argument, when $\nu \geq c_n$, $\overline{E_n^\nu} \simeq \overline{M}_1/p^n\overline{M}_1$, a free $W(\overline{k})/p^nW(\overline{k})$-module of rank 1.
As $\Lambda \xrightarrow{i} W(\overline{k})$ is faithfully flat and $\overline{E_n^\nu} = E_n^\nu \otimes_\Lambda W(\overline{k})$ is a free $W(\overline{k})/p^nW(\overline{k})$-module of rank 1 for $\nu \geq c_n$, then E_n^ν is a free $\Lambda/p^n\Lambda$-module of rank 1, hence so is E_n. Also the surjectivity of $\overline{E}_{n+1} \to \overline{E}_n$ implies the surjectivity of $E_{n+1} \to E_n$. Let $M_1 = \varprojlim_{n>0} E_n$, it is easy to see that M_1 is a free Λ-module. Since $M_1 \otimes_\Lambda W(\overline{k}) = \overline{M_1}$ has slope m, so does M_1.
Now we show $\nabla(M_1) \subset M_1 \otimes_\Lambda \Omega_\Lambda$. Here $\Omega_\Lambda = \varprojlim_n \Omega^1_{(\Lambda/p^n\Lambda)/\mathbb{Z}}$ is the p-adic module of differentials. Let $\{e_1, ..., e_r\}$ be a basis of M and $e_1 \in M_1$. Suppose $\nabla(e_1) = \sum_{i=1}^r e_i \otimes \eta^i$. We need to show $\eta^i = 0$ for $i > 1$. As F^ν is a horizontal σ^ν-linear self map for $\nu > 0$, it exchanges with ∇ in the following sense: $\widetilde{F^\nu} \circ \nabla = \nabla \circ F^\nu$, where $\widetilde{F^\nu} = F^\nu \otimes \widetilde{\sigma^\nu}$ is the endomorphism of $M \otimes_\Lambda \Omega_\Lambda$ and $\widetilde{\sigma^\nu}$ is the map $\Omega_\Lambda \to \Omega_\Lambda$ given by $\alpha d\beta \mapsto \sigma^\nu(\alpha)d\sigma^\nu(\beta)$. Then from $\widetilde{F^\nu} \circ \nabla(e_1) = \nabla \circ F^\nu(e_1)$ we deduce that

$$p^{m\nu}\mu_\nu \sum_{i>1} e_i \otimes \eta^i = \sum_{i>1} F^\nu(e_i) \otimes \widetilde{\sigma^\nu}(\eta^i) \bmod M_1 \otimes \Omega_\Lambda,$$

where $\mu_\nu \in \Lambda^*$. By [3, 1.4.3], $F^\nu(M/M_1) \subset p^{m\nu+1}(M/M_1)$ for $\nu >> 0$. By comparing terms before e_i in the above equation, we get $\eta^i \in p\Omega_\Lambda$. Replace η^i by $p\widetilde{\eta^i}$ on the right side, then we get $\eta^i \in p^2\Omega_\Lambda$. By repeating, $\eta^i \in p^n\Omega_\Lambda$ for every n. By [5, 1.3.1 Proposition], Ω_Λ is a free Λ-module. Then $\eta^i = 0$ for $i > 1$. Hence M_1 is preserved by ∇. $\square$

Remark 2.5. The proposition can be generalized in the following way: Let $(\mathcal{E}, F)$ be a crystal over $S = \mathrm{Spec}(k)$. If the first break point of $NP(S, \mathcal{E})$ is $(\mu_1, \mu_1\lambda_1)$, where λ_1 is the lowest Newton slope and μ_1 is its multiplicity, then there is a unique sub-crystal $\mathcal{E}' \subset \mathcal{E}$ of rank μ_1 with its Newton slopes all equal to λ_1.
Applying the lemma to $(\wedge^{\mu_1}\mathcal{E}, \wedge^{\mu_1}F)$, we obtain a subcrystal $\mathcal{E}_1$ of $\wedge^{\mu_1}\mathcal{E}$. To see that $\mathcal{E}_1$ is of the form $\wedge^{\mu_1}\mathcal{E}'$ for some subcrystal $\mathcal{E}' \subset \mathcal{E}$, we need to use the Plücker coordinate and check if $\mathcal{E}_1$ satisfies the Plücker equations. By extending the scalars to the fraction field K of $W(\overline{k})[X]/(X^N - p)$ for some proper N, we obtain that $\mathcal{E} \otimes K$ admits a K-basis over which the matrix of F is diagonalized, hence the unique subcrystal $\mathcal{E}_1 \otimes K$ of rank 1 and slope m satisfies the Plücker equations, and so does $\mathcal{E}_1$.

3. Facts about Fundamental Groups

3.1. Let X be a noetherian normal integral scheme with its generic point η. Let $\overline{\eta}$ be a geometric point over η. By [12, Exposé V, Proposition 8.2], the canonical map $\phi : \mathrm{Gal}(\overline{K}/K) \to \pi_1(X, \overline{\eta})$ is surjective, and the kernel is $\mathrm{Gal}(\overline{K}/M)$, where $\overline{K}$ is some algebraic closure of the fraction field K of X and M is the union of all finite subextensions $K \subset L (\subset \overline{K})$ such that L is unramified over X, which means that the normalization of X in L is unramified over X.

This section focuses on describing the kernel of ϕ in terms of local kernels. Assume that the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ at every point x is an integral domain. Denote the fraction field and residue field of $\widehat{\mathcal{O}}_{X,x}$ by K_x or $k(x)$ respectively. Let $\overline{K}_x$ be an algebraic closure of K_x and $\overline{\eta}_x$ be the geometric point defined by Spec $\overline{K}_x \to$ Spec $\widehat{\mathcal{O}}_{X,x}$. Fix some injection $\omega : \overline{K} \to \overline{K}_x$ such that we have the following commutative diagram:

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \overline{K} \arrow[d, "\omega"] \\ K_x \arrow[r] & \overline{K}_x \end{tikzcd}$$

Thus we have maps of Galois groups depending on ω:

$$\psi_x : \mathrm{Gal}(\overline{K}_x/K_x) \to \mathrm{Gal}(\overline{K}/K), \quad \phi_x : \mathrm{Gal}(\overline{K}_x/K_x) \to \pi_1(\mathrm{Spec}\ \widehat{\mathcal{O}}_{X,x}, \overline{\eta}_x)$$

Proposition 3.2. *Let X be a noetherian normal integral scheme with K as its function field. Let ϕ, ϕ_x and ψ_x be the same as above. If assuming that the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ at each closed point $x \in X$ is a normal domain and that the same condition holds for the normalization of X in every finite separable extension L/K, then Ker $\phi = H$, where H is the normal closed subgroup of Gal$(\overline{K}/K)$ generated by $\{\psi_x(\text{Ker } \phi_x) |\ x$ is a closed point of $X\}$.*

Note that if moreover X is an excellent scheme, the conditions on the local ring $\mathcal{O}_{X,x}$ at every closed point $x \in X$ are satisfied. In the following, let $x \in X$ be a closed point and L/K be a finite separable subextension in $\overline{K}$ if no other description is given.

3.3. Let $\widetilde{X}$ be the normalization of X and $\widetilde{\mathcal{O}}_{X,x}$ be the integral closure of $\mathcal{O}_{X,x}$ in L. By [7, Chapter I, Proposition 8], $\widetilde{X} \to X$ is a finite morphism, and $\widetilde{\mathcal{O}}_{X,x}$ is a finitely generated $\mathcal{O}_{X,x}$-module. Let $\{x_i \in \widetilde{X},\ i \in I\}$ be the set of points over x. Since $\widetilde{\mathcal{O}}_{X,x}$ is a semilocal ring and a finite $\mathcal{O}_{X,x}$-module, then by [9, Chapter I, Theorem 4.2], $\widetilde{\mathcal{O}}_{X,x} \otimes_{\mathcal{O}_{X,x}} \widehat{\mathcal{O}}_{X,x} = \prod_{i\in I} \widehat{\mathcal{O}}_{\widetilde{X},x_i}$, where $\widehat{\mathcal{O}}_{\widetilde{X},x_i}$ is the completion of the local ring of $x_i \in \widetilde{X}$, and $\widehat{\mathcal{O}}_{\widetilde{X},x_i}$ is a finite $\widehat{\mathcal{O}}_{X,x}$-algebra. Thus we have the following cartesian diagram:

$$\begin{tikzcd} \mathrm{Spec}\ \widehat{\mathcal{O}}_{X,x} \arrow[d] & \mathrm{Spec}\ \prod_{i\in I} \widehat{\mathcal{O}}_{\widetilde{X},x_i} \arrow[l] \arrow[d] & \mathrm{Spec}\ \widehat{\mathcal{O}}_{\widetilde{X},x_i} \arrow[l] \arrow[d] \\ \mathrm{Spec}\ \mathcal{O}_{X,x} & \mathrm{Spec}\ \widetilde{\mathcal{O}}_{X,x} \arrow[l] & \mathrm{Spec}\ \mathcal{O}_{\widetilde{X},x_i} \arrow[l] \end{tikzcd}$$

3.4. By [9, Chapter I, Proposition 3.5], L is unramified over X if and only if $\Omega^1_{\widetilde{X}/X} = 0$. As the branch locus where $\Omega^1_{\widetilde{X}/X} \neq 0$ is a closed subset, and Ω^1

behaves well with respect to base change, then L is unramified over X if and only if

$$\Omega^1_{\mathrm{Spec}\ \mathcal{O}_{\widetilde{X},x_i}/\mathrm{Spec}\ \mathcal{O}_{X,x}} = 0, \text{ for every closed point } x_i \in \widetilde{X} \text{ over } x \in X.$$

As $\widehat{\mathcal{O}}_{\widetilde{X},x_i}$ is a faithfully flat $\mathcal{O}_{\widetilde{X},x}$-module, this is also equivalent to

$$\text{(3.4.1)} \quad \Omega^1_{\mathrm{Spec}\ \widehat{\mathcal{O}}_{\widetilde{X},x_i}/\mathrm{Spec}\ \widehat{\mathcal{O}}_{X,x}} = 0, \text{ for every closed point } x_i \in \widetilde{X} \text{ over } x \in X.$$

3.5. Let $L.K_x = \omega(L)K_x$. It is clear that $L.K_x$ is a separable extension of K_x. Actually $L.K_x$ is the fraction field of $\widehat{\mathcal{O}}_{\widetilde{X},x_i}$ for some i. Since by base change the diagram in 3.3 from Spec $\widehat{\mathcal{O}}_{X,x} \to$ Spec $\mathcal{O}_{X,x}$ to Spec $K_x \to$ Spec K, we have $L \otimes_K K_x \simeq \prod_{i\in I} Frac(\widehat{\mathcal{O}}_{\widetilde{X},x_i})$; by choosing ω, one has to choose $L \to Frac(\widehat{\mathcal{O}}_{\widetilde{X},x_i})$. Hence L/K is fixed by H if and only if for every closed point $x \in X$, there exists some x_i over x such that $Frac(\widehat{\mathcal{O}}_{\widetilde{X},x_i})$ is unramified over Spec $\widehat{\mathcal{O}}_{X,x}$; by assumption this is equivalent to Spec $\widehat{\mathcal{O}}_{\widetilde{X},x_i} \to$ Spec $\widehat{\mathcal{O}}_{X,x}$ being unramified. Hence L/K is fixed by H if and only if
(3.5.1)

$$\Omega^1_{\mathrm{Spec}\ \widehat{\mathcal{O}}_{\widetilde{X},x_i}/\mathrm{Spec}\ \widehat{\mathcal{O}}_{X,x}} = 0, \text{ for every closed point } x \in X, \text{ some } x_i \in \widetilde{X} \text{ over } x.$$

3.6. Assume that L/K is a finite Galois extension. Let $U \subset X$ be an affine neighborhood of x and $\widetilde{U}$ be its normalization in L. By [8, Chapter 2, 5.E], for two given points $x_i, x_j \in \widetilde{U}$ over $x \in U$, there exists a U-automorphism of $\widetilde{U}$ mapping x_i to x_j, hence $\Omega^1_{\mathrm{Spec}\ \mathcal{O}_{\widetilde{U},x_i}/\mathrm{Spec}\ \mathcal{O}_{U,x}} \simeq \Omega^1_{\mathrm{Spec}\ \mathcal{O}_{\widetilde{U},x_j}/\mathrm{Spec}\ \mathcal{O}_{U,x}}$. It follows that $\widetilde{X} \to X$ is unramified at some point over x if and only if it is unramified at every point over x.

Proof of Proposition 3.2: Let N be the subfield of $\overline{K}$ fixed by H. Since both H and Ker ψ are normal subgroups, then it suffices to show that $N = M$. Assume L/K is a finite Galois subextension. From discussions in 3.6, the conditions 3.4.1 and 3.5.1 are equivalent. Then we have $L \subset M \Leftrightarrow L \subset N$, hence $M = N$. □

We will also need the following facts about Galois groups.

Claim 3.7. Let (K, ν) be a henselian field with a nonarchimedean valuation ν. Let (K_ν, ν) be its completion. Denote by $\overline{K}$(resp. $\overline{K}_\nu$) the algebraic closure of K(resp. K_ν). Then the homomorphism Gal $(\overline{K}_\nu/K_\nu) \to$ Gal $(\overline{K}/K)$ is surjective.

Fact 3.8. *Let l/k be an algebraic extension. Let $K = k((t))$, $\widehat{L} = l((t))$, and $L = \bigcup m((t))$ where m runs over all finite subextensions of k in l. There is an obvious valuation ν on L and $\widehat{L}$ by sending t^n to n. It is clear that $\widehat{L}$ is the completion of L. As the valuation ring of L is $R = \bigcup m[[t]]$, from*

Definition(6.1) in [13, Chapter II]*, L is a henselian field. Thus $Gal_{\widehat{L}} \to Gal_L$ is surjective.*

Corollary 3.9. *Let R be a discrete valuation ring of characteristic p with fraction field K and residue field k. let $s \in$ Spec R be the closed point and $\overline{\eta}$ be a geometric point over the generic point. Let R_s be the completion of R, which is of the form $k[[t]]$. Then the kernel of the canonical homomorphism $Gal_K \xrightarrow{\phi} \pi_1(\text{Spec } R, \overline{\eta})$ is the normal subgroup of Gal_K generated by the image of the composition $Gal_{\overline{k}((t))} \longrightarrow Gal_{k((t))} \xrightarrow{\psi_s} Gal_K$.*

Proof: Since Spec R satisfies the assumption in Proposition 3.2, Ker ϕ is generated by $\psi_s(\text{Ker } \phi_s)$. Apply Fact 3.8 to the case when $l = k^{sep}$. Note that L is the maximal unramified algebraic extension of $X = \text{Spec } k[[t]]$ in the sense of 3.1, and hence Ker ϕ_s in 3.1 is the normal subgroup generated by the image of $Gal_{k^{sep}((t))} \twoheadrightarrow \text{Gal}_L \to \text{Gal}_{k((t))}$. Apply Fact 3.8 to the field extension $\overline{k}/k^{sep}$ to see that $\text{Gal}_{\overline{k}((t))} \to \text{Gal}_{k^{sep}((t))}$ is surjective. In conclusion, Ker ϕ_s is the image of $\text{Gal}_{\overline{k}((t))} \to \text{Gal}_{k((t))}$. □

4. Galois Representations Associated to F-crystals of Rank 1

4.1. Consider an F-crystal $\mathcal{E}$ of rank 1 and slope m over k. Let (M, ∇, F) over Λ be the triple defining the crystal $\mathcal{E}$. If $\{e\}$ is chosen to be the basis of M, then $F(e) = p^m \mu e$, where μ is a unit in $\Lambda \subset W(k^{pf})$. By 2.3, there exists some unit $\alpha \in W(\overline{k})$ such that $F(e \otimes \alpha) = p^m e \otimes \alpha$, i.e. $\sigma(\alpha)\mu = \alpha$. As every $g \in \text{Gal}(\overline{k}/k) = \text{Gal}(\overline{k}/k^{pf})$ can be uniquely lifted as a $W(k^{pf})$-automorphism of $W(\overline{k})$, it is easy to show that $g(\alpha)\alpha^{-1} \in \mathbb{Z}_p^*$. Thus we get a continuous homomorphism $\rho : \text{Gal}(\overline{k}/k) \to \mathbb{Z}_p^*$ by sending g to $g(\alpha)\alpha^{-1}$.

Definition 4.2. Let $\mathcal{E}$ be an F-crystal over a noetherian integral scheme X of characteristic p. Let K be the fraction field of X and η be the generic point. Assume that the first break point of $NP(X, \mathcal{E})_\eta$ is $(1, m)$, where $m \in \mathbb{Z}_{\geq 0}$. Then by Proposition 2.4, there exists a unique subcrystal $\mathcal{E}_1 \subset \mathcal{E}_\eta$ of rank 1 and slope m. By the above discussion we obtain from the crystal $\mathcal{E}_1$ a continuous homomorphism $\rho : \text{Gal}(\overline{K}/K) \to \mathbb{Z}_p^*$. We call it *the Galois representation associated to* $\mathcal{E}$, or *the associated representation of* $\mathcal{E}$.

4.3. Let X and Y be noetherian integral schemes. Let $f : X \to Y$ be a morphism mapping the generic point of X to the generic point of Y. Assume that $\mathcal{E}$ is a crystal over Y satisfying the assumption of the definition. Then the representation associated to $\mathcal{E}|_X$ is the composition $\text{Gal}_{K(X)} \to \text{Gal}_{K(Y)} \to \mathbb{Z}_p^*$.

Lemma 4.4. *Let $(\mathcal{E}, F_1)$ and $(\mathcal{E}', F_2)$ be two F-crystals over a noetherian integral scheme X satisfying the assumptions in Definition 4.2. If there exists an isogeny $\psi : \mathcal{E} \to \mathcal{E}'$, then their associated representations are identical.*

Proof: Let $\mathcal{E}_1$ (resp. $\mathcal{E}'_1$) be the subcrystal of $(\mathcal{E})_\eta$ (resp. $(\mathcal{E}')_\eta$) obtained in Proposition 2.4. As $\psi \circ F_1 = F_2 \circ \psi$, then $\psi(\mathcal{E}_1) \subset \mathcal{E}'_1$. Actually we can

choose a basis e_i of $\mathcal{E}_1$ (resp. $\mathcal{E}_1'$) so that $\psi(e_1) = p^n e_2$ for some $n \in N$, and $F_i e_i = p^m \mu e_i$ for some unit $\mu \in \Lambda$. Then it is obvious that the Galois representations associated to $\mathcal{E}$ and $\mathcal{E}'$ are identical. □

LEMMA 4.5. *Let $\mathcal{E}$ be an F-crystal of rank 1 and slope $m \in \mathbb{Z}_{\geq 0}$ over $S =$ Spec k, where k is a field of characteristic p. If the associated representation is trivial, then $\mathcal{E}$ is a trivial crystal, i.e. there exists some basis $\{e\}$ of $\mathcal{E}$ such that $F(e) = p^m e$, and $\nabla(e) = 0$.*

Proof: Let e be a basis of $\mathcal{E}$, and $F(e) = p^m \mu e$. From 4.1, there is some unit $\alpha \in W(\overline{k})$ such that $\sigma(\alpha)\mu = \alpha$; the associated representation is trivial if and only if the unit $\alpha \in W(k^{pf})$. It suffices to show that $\alpha \in \Lambda$, and $\nabla(e) = 0$ follows automatically.
Let $U^n(k) = 1 + p^n \Lambda$ and $U^n(k^{pf}) = 1 + p^n W(k^{pf})$. First choose $\mu \in U^1(k)$ and $\alpha \in U^1(k^{pf})$. Considering $\sigma(\alpha)\mu = \alpha(\bmod p)$, we have $\overline{\alpha}^p \overline{\mu} = \overline{\alpha}$, where $\overline{\alpha} = (\alpha \bmod p) \in k^{pf}$. Moreover, the equation implies that $\overline{\alpha}$ is separable over k, and hence $\overline{\alpha} \in k$. Choose $\gamma_0 \in \Lambda$ such that $\gamma_0(\bmod p) = \overline{\alpha}$. Replace the basis e by $\gamma_0 e$, then replace μ by $\sigma(\gamma_0)\mu\gamma_0^{-1}$ and α by $\alpha \cdot \gamma_0^{-1}$. Then $\sigma(\alpha)\mu = \alpha$ still holds, and $\mu \in U^1(k)$, $\alpha \in U^1(k^{pf})$.
The induction step: Assume $\mu_{n-1} \in U^n(k)$, $\alpha_{n-1} \in U^n(k^{pf})$, and $\sigma(\alpha_{n-1})\mu_{n-1} = \alpha_{n-1}$. It suffices to show that there exists some $\gamma_n \in U^n(k)$ such that $\gamma_n = \alpha_{n-1}(\bmod p^{n+1})$. Write $\mu_{n-1} = 1 + p^n \nu_n$, $\alpha_{n-1} = 1 + p^n \delta_n$ for some $\nu_n \in \Lambda$, $\delta_n \in W(k^{pf})$. By assumption we have

$$\sigma(\delta_n) + \nu_n = \delta_n(\bmod p) \text{ or } \overline{\delta}_n^p + \overline{\nu}_n = \overline{\delta}_n$$

As $\overline{\delta}_n \in k^{pf}$, and since the above equation implies that it is separable over k, $\overline{\delta}_n \in k$. Hence we can choose $\gamma_n \in U^n(k)$ such that $\gamma_n = \alpha_{n-1}(\bmod p^{n+1})$.
Then let $\mu_n = \sigma(\gamma_n)\mu_{n-1}\gamma_n^{-1}$ and $\alpha_n = \alpha_{n-1} \cdot \gamma_n^{-1}$. We can easily see that they satisfy the induction assumptions. Thus we can get a sequence $\{\gamma_n \in U^n(k) | n \geq 1\}$. As Λ is complete, $\prod_n \gamma_n$ converges to $\beta \in \Lambda$. It is not hard to see that $\alpha \cdot \beta^{-1} = 1$, and thus $\alpha \in \Lambda$. □

PROPOSITION 4.6. *Let R be a discrete valuation ring of characteristic p with fraction field K and residue field k. Let $\mathcal{E}$ be an F-crystal over Spec R. Let η and s be the generic and closed point of Spec R. Assume that the first break point of $NP(\mathcal{E})_\eta$ is $(1, m)$. Then the following two conditions are equivalent:*
(a) the Galois representation associated to $\mathcal{E}$ is unramified, i.e., it factors through $\phi : Gal_K \to \pi_1(\text{Spec } R)$.
(b) the first break point of $NP(\mathcal{E})_s$ is $(1, m)$.

Proof: First consider Spec $\overline{k}[[t]] \to$ Spec R. By Corollary 3.9 and 4.3, Condition (a) is equivalent to the triviality of the associated representation of $\mathcal{E}|_{\text{Spec } \overline{k}[[t]]}$. Moreover, as the Newton polygons of E are preserved after pulled back to Spec $\overline{k}[[t]]$, Condition (b) holds if and only if the first break point of $NP(\mathcal{E}|_{\text{Spec } \overline{k}[[t]]})_s$ is $(1, m)$. Hence it suffices to prove the proposition for $R = k[[t]]$ with k algebraically closed. Note that in this case (a) is equivalent to the following (a)' the Galois representation associated to $\mathcal{E}$ is trivial.

Condition (b)$\Rightarrow$ (a)':by [3, Corollary 2.6.2], $\mathcal{E}$ is isogenous to an F-crystal $\mathcal{E}'$ which is divisible by p^m, which contains a subcrystal $\mathcal{E}'_1$ of rank 1 and slope m. By Lemma 4.4, the Galois representation in question is the same as the one associated to $\mathcal{E}'_1|_{\text{Spec } K}$. By [3, Theorem 2.7.4], $\mathcal{E}'_1$ becomes isogenous to a constant F-crystal over $k((t))^{pf}$, and therefore the associated representation is trivial.
(a)'$\Rightarrow$(b): By Lemma 4.5, $\mathcal{E}_{\text{Spec } K}$ has a trivial subcrystal of rank 1 and slope m. Then we get an injection $\Phi : \mathcal{L}_{\text{Spec}K} \to \mathcal{E}_{\text{Spec } K}$, where $\mathcal{L}$ is a trivial F-crystal of rank 1 and slope m over Spec R. Apply [2, Main Theorem] to $\mathcal{E}, \mathcal{L}$ and Φ. We obtain a nontrivial map $\mathcal{L} \to \mathcal{E}$. Restricting to s,we see that $\mathcal{E}_s$ contains a subcrystal of rank 1 and slope m. On the other hand, by Grothendieck's specialization theorem [3, 2.3.1], $NP(\mathcal{E})_s$ lies on or above $NP(\mathcal{E})_\eta$. Hence $(1, m)$ is the first break point of $NP(\mathcal{E})_s$. $\square$

5. The proof

Assume the common break point is $P = (i, m)$. If we assume that $\operatorname{codim}(U, \{s\}^-) > 1$, then we just need to show that P is also a break point of $NP(\mathcal{E})_s$.
Step 1: Reduce to the special case when the common break point P is of the form $(1, m)$.
In the general case, let $\mathcal{E}' = \wedge^i \mathcal{E}$. By assumption, $(1, m)$ is the first break point of $NP(\mathcal{E}')_x$ for all $x \in U \backslash \{s\}$. Applying the result for the special case, we obtain that $(1, m)$ is a break point of $NP(\mathcal{E}')_s$, and hence P is a break point of $NP(\mathcal{E})_s$.
Step 2: First as S is locally noetherian, we may shrink S to an open affine neighborhood Spec A of s such that (Spec $A \backslash \{s\}) \subset U$. Then we follow the same reduction steps as in the proof of [1, Theorem 4.1]. We obtain that there exists a Noetherian complete local normal domain A of dimension 2 with algebraically closed residue field k and a morphism ϕ : Spec $A \to S$ that maps closed point to s and other points into U. Hence it suffices to prove the statement when S is the spectrum of a Noetherian complete local normal domain A of dimension 2 with algebraically closed residue field k, s is the closed point and $U = S \backslash \{s\}$.
Up to now, we have shown that it suffices to prove the following simplified statement: *let A be a Noetherian complete local normal integral domain of dimension 2 with algebraically closed residue field k. Let $s \in S =$ Spec A be the closed point and $U = S \backslash \{s\}$. If $(1, m)$ is the first break point of $NP(\mathcal{E})_x$ for every $x \in U$, then $(1, m)$ is the first break point of $NP(\mathcal{E})_s$.*
Let K be the fraction field of A. Consider the Galois representation ρ : $\operatorname{Gal}(\overline{K}/K) \to \mathbb{Z}_p^*$ defined in 4.2. Let H be the kernel of the composition of ρ and $\mathbb{Z}_p^* \xrightarrow{\text{mod } p} \mathbb{F}_p^*$. Let L be the subfield of $\overline{K}$ fixed by H. As $\operatorname{Gal}(\overline{K}/K)/H$ is a finite set, L is a finite Galois extension of K. Let $\widetilde{A}$ be the integral closure of A in L. By a standard argument, we see that $\widetilde{A}$ is a Noetherian complete

local normal domain of dimension 2 with residue field k. Consider the finite morphism Spec $\widetilde{A} \to$ Spec A. It is not hard to see that we only need to prove the statement for $\widetilde{A}$. Replacing A by $\widetilde{A}$, we may assume that $\mathrm{Im}(\rho) \subset 1 + p\mathbb{Z}_p$ and that the homomorphism $\mathrm{Gal}(\overline{K}/K) \xrightarrow{\rho} \mathbb{Z}_p^* \xrightarrow{\log} \mathbb{Z}_p$ is valid.

Let $x \in U$. Assume that ϕ, ϕ_x and ψ_x are the same as in 3.2. Since $\mathcal{E}|_{\mathrm{Spec}\ \widehat{\mathcal{O}}_{U,x}}$ satisfies the assumptions and Condition (b) in Proposition 4.6, the associated representation of $\mathcal{E}|_{\mathrm{Spec}\ \widehat{\mathcal{O}}_{U,x}}$, which is the composition of ψ_x and ρ, factors through ϕ_x. It follows that $\psi_x(\mathrm{Ker}\phi_x) \subset \mathrm{Ker}\rho$. By Proposition 4.6, we obtain that ρ factors through ϕ. Thus we obtain a map $\iota : \pi_1(U, \overline{\eta}) \to \mathbb{Z}_p^* \to \mathbb{Z}_p$.

Take a resolution of singularities $\widetilde{S} \to S$; if A happens to be regular, let $\widetilde{S}$ be the blowup of the special point of Spec A. Then the main result of [1, Section 3] implies that ι can be extended to $\widetilde{\iota} : \pi_1(\widetilde{S}, \overline{\eta}) \to \mathbb{Z}_p$. Let ξ denote the generic point of a component of the exceptional fibers of $\widetilde{S} \to S$. Now we have the following diagrams:

$$\begin{tikzcd}
\mathrm{Spec}\ K \arrow[r] \arrow[d] & U \arrow[d] \\
\mathrm{Spec}\ \mathcal{O}_{\widetilde{S},\xi} \arrow[r] & \widetilde{S}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Gal}(\overline{K}/K) \arrow[r] \arrow[d] \arrow[drr, dotted] & \pi_1(U, \overline{\eta}) \arrow[d] & \\
\pi_1(\mathrm{Spec}\ \mathcal{O}_{\widetilde{S},\xi}, \overline{\eta}) \arrow[r] & \pi_1(\widetilde{S}, \overline{\eta}) \arrow[r] & \mathbb{Z}_p^*
\end{tikzcd}$$

By definition the representation associated to $\mathcal{E}_{\mathrm{Spec}\ \mathcal{O}_{\widetilde{S},\xi}}$ is the dotted arrow, and it is unramified by the above commutative diagram. By Proposition 4.6 again, $(1, m)$ is the first break point of $NP(\widetilde{S}, \mathcal{E})_\xi$. Since ξ is mapped to s, $(1, m)$ is thus the first break point of $NP(\mathcal{E})_s$. $\square$

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